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Albert-Ludwigs-Universität Freiburg
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Modelltheorie II

Homework Sheet 8

Deadline: 26.06.2023, 14 Uhr

Unless explicitly mentioned, we work inside a sufficiently large saturated model $\mathbb{U}$ of an ω-stable complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

Exercise 1 (6 Points). Consider a subset C of $\mathbb{U}$ and a C-indiscernible sequence $(a_n)_{n\in\mathbb{N}}$ which is also C-independent.

a) Given $B \supset C$ with $B \mathop{\smile\!\!\!\!\!\!\mid}_C \{a_n\}_{n\in\mathbb{N}}$, show that $a_1, \ldots, a_n \equiv_B a_{m_1}, \ldots, a_{m_n}$ for all pairwise distinct $m_1, \ldots, m_n$.

b) Does the above hold in the random graph, setting $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $A \cap B \subset C$?

Exercise 2 (7 Points). Consider a definable group G acting definably on a definable set X.

a) Given an arbitrary (non-empty) subset S of X, show that there are $s_1, \ldots, s_n$ in S such that for all $g \in G$

$$g \star s = s \text{ for all } s \text{ in } S \iff g \star s_i = s_i \text{ for } i = 1, \ldots, n.$$

b) Compute the rank $\mathrm{RM}(\mathrm{GL}_n(\mathbb{C}))$ in the strongly minimal field $\mathbb{C}$, where $\mathrm{GL}_n(\mathbb{C})$ is the subset of the square $n \times n$-matrices $\mathrm{Mat}_{n\times n}(\mathbb{C})$ consisting of the regular ones.

c) Let now S be the collection of strictly upper triangular matrices in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ (so each element of S is nilpotent!). Find an explicit set of elements s_i as in (a) for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Mat}_{n\times n}(\mathbb{C})$ by conjugation.

Exercise 3 (7 Points).

a) Consider a fixed algebraic closure K^{alg} of a field K of characteristic different from 2. Show that the number of intermediate fields $K \subset L \subset K^{alg}$ of degree $[L : K] \leq 2$ equals the index $(K^* : (K^*)^2)$ of the squares.

Consider now an ultraproduct $K = \prod_{\mathcal{U}} \mathbb{F}_p$ with respect to a non-principal ultrafilter $\mathcal{U}$ on the set P of prime numbers, where we view each finite field $\mathbb{F}_p$ as an $\mathcal{L}_{ring}$-structure.

b) Determine the characteristic of K and show that the underlying additive group $(K, +)$ is connected.

c) Is the underlying multiplicative group $(K^*, \cdot)$ connected?

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.