Prof. Amador Martin-Pizarro
Übungen: Dr. Francesco Gallinaro

Albert-Ludwigs-Universität Freiburg
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Modelltheorie II

Homework Sheet 6

Deadline: 12.06.2023, 14 Uhr

Exercise 1 (6 Points). Consider a sufficiently large saturated model $\mathbb{U}$ of an ω-stable complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

a) Given a fixed $\mathcal{L}$-formula $\varphi(x, y)$ and an elementary substructure $\mathcal{M}$ of $\mathbb{U}$, show that for a fixed type p over M, any two φ-definitions are logically equivalent.

We now assume that T is the theory of an equivalence relation with infinitely many classes whose classes are all infinite.

b) Given an elementary substructure $\mathcal{M}$ of $\mathbb{U}$ and an n-type p over M, describe explicitly its defining scheme $d_p\varphi$, as the formula φ varies among all atomic formulae.

c) If $\mathcal{M} = \mathbb{U}$, describe explicitly the canonical base of p.

Exercise 2 (3 Points). Consider the ordered set $\mathbb{Q}$ as a model of the theory DLO of dense linear orders without endpoints. Which ones among the following types over $\mathbb{Q}$ are definable?

- The unique completion of $\{n < x\}_{n\in\mathbb{N}}$.
- The unique completion of $\{0 < x < \frac{1}{n}\}_{1\leq n\in\mathbb{N}}$.
- The type $\mathrm{tp}(\sqrt{2}/\mathbb{Q})$.

Exercise 3 (6 Points). Consider an indiscernible sequence $(a_n)_{n\in\mathbb{N}}$ of tuples of some fixed length of some sufficiently large saturated model $\mathbb{U}$ of an ω-stable complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

a) Show that the collection of $\mathbb{U}$-instances

$$\mathrm{Av}((a_n)_{n\in\mathbb{N}}) = \{\varphi[x, b] \mid b \in \mathbb{U} \ \&\ \mathbb{U} \models \varphi[a_n, b] \text{ for all but finitely many } n\}$$

is a global (complete) type.

Hint: Build a sequence witnessing OP for φ using Compactness.

b) Show that the canonical base of the above global type is contained in $\mathrm{dcl}^{eq}(\{a_n\}_{n\in\mathbb{N}})$.

Exercise 4 (5 Points). Consider a sufficiently large cardinal $\kappa \geq \aleph_0$ and a strongly κ-homogeneous κ-saturated model $\mathbb{U}$ of a complete first-order theory T with infinite models in a fixed language $\mathcal{L}$. Given a type p over A and a subset C of A, we say that p is *coheir* over C if every instance $\varphi(x, a)$ in p has a realization in C. Notice that every type over an elementary substructure $\mathcal{M}$ of $\mathbb{U}$ is a coheir over M.

a) Give an example of a type over A which is not coheir over A.

b) Assume that p in $\mathrm{S}(A)$ is a coheir over C. Show that for every $B \supset A$, there exists an extension q of p over B which is coheir over C.

c) Give an explicit example of tuples a and b as well as an elementary substructure $\mathcal{M}$ of a suitable theory T such that $\mathrm{tp}(a/M, b)$ is coheir over M but $\mathrm{tp}(b/M, a)$ is not.

Hint: Exercise 2.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.