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Übungen: Dr. Francesco Gallinaro — Sommersemester 2023

Modelltheorie II
Homework Sheet 4
Deadline: 22.05.2023, 14 Uhr

Exercise 1 (4 Points). Show that the theory DLO of dense linear orders without endpoints eliminates finite imaginaries.

Exercise 2 (10 Points). For $\kappa \geq \aleph_0$ sufficiently large, consider a κ-saturated κ-strongly homogeneous model $\mathbb{U}$ of a complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

a) Given a real element a and a subset B of real parameters in $\mathbb{U}$ such that a is algebraic over B in the corresponding expansion $\mathbb{U}^{eq}$ as an $\mathcal{L}^{eq}$-structure, show that the $\mathcal{L}$-type $\mathrm{tp}(a/B)$ contains an algebraic formula.

b) Suppose now that f is a partial elementary map with $|\mathrm{Dom}(f)| < \kappa$. Given an imaginary element α in $\mathbb{U}^{eq}$, show that there is an extension $\widehat{f}$ of f to a partial elementary map of $\mathbb{U}^{eq}$ whose domain contains α.

   Deduce that the groups $\mathrm{Aut}(\mathbb{U})$ and $\mathrm{Aut}(\mathbb{U}^{eq})$ are isomorphic.

c) Let X an infinite interpretable subset (for some finite cartesian product of sorts in $\mathbb{U}^{eq}$). Show that $|X(\mathbb{U}^{eq})| \geq \kappa$, without assuming that compactness holds in many-sorted logic.

We assume now that T is the theory of the field of real numbers in the ring language.

d) Consider the following equivalence relation on $\mathbb{U}$:

$$x \simeq y \Longleftrightarrow 0 \leq |x-y| < \frac{1}{n} \text{ for all } n \text{ in } \mathbb{N}.$$

   Show that $\simeq$ is not definable.

e) If we now restrict $\simeq$ to the definable interval $X = [0,1]$, what is the cardinality of $X(\mathbb{U})/\simeq$?

Exercise 3 (6 Points). For $\kappa \geq \aleph_0$ sufficiently large, consider a κ-saturated κ-strongly homogeneous model $\mathbb{U}$ of a complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

a) Given an $\mathcal{L}$-substructure $\mathcal{M}$ of $\mathbb{U}$, denote by $\mathcal{M}^{eq}$ the collection of imaginaries with a representative in M. Show that $\mathcal{M}^{eq}$ is an $\mathcal{L}^{eq}$-substructure of $\mathbb{U}^{eq}$.

b) Show that $\mathcal{M}^{eq}$ is an elementary substructure of the model $\mathbb{U}^{eq}$ of T^{eq} if $\mathcal{M} \preceq \mathbb{U}$.

c) Given a subset A of $\mathbb{U}$, show (without assuming compactness in many-sorted logic) that

$$\mathrm{acl}^{\mathrm{eq}}(A) = \bigcap_{\substack{\mathcal{M} \preceq \mathbb{U} \\ A \subset \mathcal{M}}} \mathcal{M}^{eq}.$$

   Hint: Write a suitable partial type in infinitely many real variables.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.