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Modelltheorie II

Homework Sheet 3

Deadline: 15.05.2023, 14 Uhr

Exercise 1 (4 Points). Consider a sufficiently saturated ambient model $\mathbb{U}$ of a complete first-order theory with infinite models in a fixed language $\mathcal{L}$.

Given an instance $\varphi[\bar{x}, \bar{a}]$ with parameters in $\mathbb{U}$, assume that there are a natural number $k \geq 2$ and a sequence $(\psi_n)_{n\in\mathbb{N}}$ of definable subsets of $\varphi[\bar{x}, \bar{a}]$ with parameters in $\mathbb{U}$, each of Morley rank at least α, such that the intersection of k many instances among the ψ_n's has Morley rank strictly less than α, that is,

$$\mathrm{RM}(\bigwedge_{j=1}^{k} \psi_{n_j}) < \alpha$$

for every distinct $n_1, \ldots, n_k$. Show that $\varphi[\bar{x}, \bar{a}]$ has Morley rank at least $\alpha + 1$.

Hint: A ranked definable set is the finite disjoint union of its irreducible components.

Exercise 2 (3 Points). Consider a sufficiently saturated ambient model $\mathbb{U}$ of a complete first-order theory with infinite models in a fixed language $\mathcal{L}$. Two tuples $\bar{a}$ and $\bar{b}$ of the same length n have the same *strong type* over C if they belong to the same class with respect to every C-definable equivalence relation on $\mathbb{U}^n \times \mathbb{U}^n$ with only finitely many equivalence classes.

a) If $\bar{a}$ and $\bar{b}$ have the same strong type over C, show that $\bar{a} \equiv_C \bar{b}$.

b) Show that all elements of an indiscernible sequence $(a_n)_{n\in\mathbb{N}}$ have the same strong type over C.

Exercise 3 (6 Points). Consider the language $\mathcal{L} = \{P_n\}_{n\in\mathbb{N}}$, with each P_n a unary predicate, and the $\mathcal{L}$-Theory T of *independent predicates* whose models are exactly the $\mathcal{L}$-structures $\mathcal{A}$ such that for every collection of indexes $n_1, \ldots, n_r$ and $m_1, \ldots, m_s$ with $\{n_i\}_{i=1}^r \cap \{m_j\}_{j=1}^s = \emptyset$, the definable set

$$\bigcap_{i=1}^{r} P_{n_i}^{\mathcal{A}} \cap \bigcap_{j=1}^{s} A \setminus P_{m_j}^{\mathcal{A}}$$

is infinite.

a) Does T have Vaught pairs?

b) Does T eliminate $\exists^\infty$?

c) For which cardinals $\kappa \geq \aleph_0$ is the theory κ-stable?

d) Describe explicitly the types with ordinal-valued Morley rank over an $\aleph_0$-saturated model of T.
Hint: Prove inductively on α a certain property of formulae first.

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Exercise 4 (7 Points). The language $\mathcal{L} = \{f, P, Q, P_n \mid 2 \leq n \in \mathbb{N}\}$ consists of a unary function symbol f, as well as unary predicates P, Q and P_n for every $n \geq 2$ in $\mathbb{N}$.

We consider the $\mathcal{L}$-theory T whose models are exactly the $\mathcal{L}$-structures $\mathcal{A}$ such that the universe is a disjoint union of the infinite sets $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$; the function $f^{\mathcal{A}}$ restricted to $P^{\mathcal{A}}$ is the identity and maps $Q^{\mathcal{A}}$ surjectively onto $P^{\mathcal{A}}$; moreover, the infinite set $P_n^{\mathcal{A}}$ consists of the elements a in P whose fibre $(f^{\mathcal{A}})^{-1}(a)$ has size exactly n. Notice that T is complete and has quantifier elimination.

a) Show that T is totally transcendental.

b) Compute the Morley ranks and degrees of $P(y)$ and $Q(x)$.

c) Compute the Morley rank of $f^{-1}(a)$ where a realizes the type of maximal Morley rank in P.

d) Is Morley rank additive in T?

   Hint: $f(b) = a$.