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Übungen: Dr. Francesco Gallinaro — Sommersemester 2023

Modelltheorie II
Homework Sheet 2
Deadline: 08.05.2023, 14 Uhr

Exercise 1 (5 Points). Consider the language $\mathcal{L} = \{E\}$ with E a binary relation symbol as well as the $\mathcal{L}$-theory T whose models are exactly the $\mathcal{L}$-structures in which E is an equivalence relation with exactly one class of size n for every natural number $n \geq 1$.

a) Compute the maximal Cantor-Bendixson rank of every type in the compact Hausdorff space $S_1(\mathcal{M}_0)$, with $\mathcal{M}_0$ the prime model of T.

b) Compute the Morley rank and degree of the formula $(x \doteq x)$.

c) Consider now an $\aleph_0$-saturated model $\mathcal{M}$ of T and a type p over M of maximal Morley rank. Given a proper elementary extension $\mathcal{N}$ of $\mathcal{M}$, describe explicitly all the extensions q of p over N with $\mathrm{RM}(q) < \mathrm{RM}(p)$.

Exercise 2 (11 Points). We work inside a sufficiently saturated infinite ambient model $\mathbb{U}$ of a complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

a) Given a subset A of $\mathbb{U}$ and a type $p(\bar{x})$ over A, show that $\mathrm{CB}(p) \leq \mathrm{RM}(p)$, computing the Cantor-Bendixson rank in the compact Hausdorff space $S_{|\bar{x}|}(A)$.

b) Assume the instance $\varphi[\bar{x}, \bar{a}]$ has Morley rank α and is α-indecomposable. Show that so is $\varphi[\bar{x}, \bar{b}]$ for every realization $\bar{b}$ of $\mathrm{tp}(\bar{a})$.

Let now $\mathcal{M}$ be an $\aleph_0$-saturated elementary substructure of $\mathbb{U}$.

c) Assume now that the instance $\varphi[\bar{x}, \bar{a}]$, with $\bar{a}$ in M, has Morley rank α and Morley degree n. Show that there is a decomposition of $\varphi[\bar{x}, \bar{a}]$ into n disjoint instances with parameters in M, each α-indecomposable of Morley rank α.

d) Conclude that every type over M of Morley rank α is *stationary*, that is, of Morley degree 1.

e) Show by transfinite induction that $\mathrm{RM}(p) = \mathrm{CB}(p)$, with $\mathrm{CB}(p)$ the Cantor-Bendixson rank of p computed in $S_{|\bar{x}|}(\mathcal{M})$.

Hint: Every formula belongs to a type with the same Morley rank.

Exercise 3 (4 Points). A natural number m is a prefix of n in $\mathbb{N}$ if m consists of the first digits of n, that is, if $n = m \cdot 10^k + r$ for some $0 \leq r < 10^k$ in $\mathbb{N}$.

Consider now an infinite subset $A \subset \mathbb{N}$ closed under removing last digits, that is, whenever n belongs to A and has last digit ℓ, then $\frac{n-\ell}{10}$ is also in A. Show that A contains an infinite sequence $(a_n)_{n\in\mathbb{N}}$ of distinct elements such that a_m is a prefix of a_n if $m \leq n$.

Hint: Trees.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.