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Übungen: Dr. Francesco Gallinaro — Sommersemester 2023

Modelltheorie II
Homework Sheet 1
Deadline: 02.05.2023, 10 Uhr

Exercise 1 (6 Points).
We work inside a sufficiently saturated ambient model $\mathbb{U}$ of a complete first-order theory T in a fixed language $\mathcal{L}$. Whenever we use the word definable, it is meant with possible parameters.

Consider two definable subsets $X \subset \mathbb{U}^n$ and $Y \subset \mathbb{U}^m$ as well as a definable function $f : X \to Y$, that is, its graph

$$\mathrm{Graph}(f) = \{(\bar{x}, \bar{y}) \in \mathbb{U}^{n+m} \mid \bar{x} \in X, \bar{y} \in Y \ \& \ f(\bar{x}) = \bar{y}\}$$

is a definable subset.

a) Show that $\mathrm{RM}(X) = \mathrm{RM}(Y)$ if f is a definable bijection. Does the converse hold?

b) Assume now that there is a fixed ordinal α such that for every $\bar{y}$ in Y the fiber $f^{-1}(\bar{y})$ has Morley rank $\mathrm{RM}(f^{-1}(y)) \geq \alpha$. Show that $\mathrm{RM}(X) \geq \alpha + \mathrm{RM}(Y)$

   Hint: Show by transfinite induction on β that $\mathrm{RM}(X) \geq \alpha + \beta$ whenever $\mathrm{RM}(Y) \geq \beta$.

Exercise 2 (14 Points).
Given a compact topological space X, define its *Cantor-Bendixson derivative* as follows:

$$\Gamma(X) = \{x \in X \mid x \text{ is not an isolated point of } X\}.$$

a) Show that $\Gamma(X)$ is a closed subset of X. In particular, the set $\Gamma(X)$ equipped with the subspace topology is again compact.

Define now $\Gamma^\alpha(X) = \begin{cases} X, \text{ if } \alpha = 0 \\ \Gamma(\Gamma^\beta(X)), \text{ if } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} \Gamma^\beta(X), \text{ if } \alpha \text{ is a limit ordinal} \end{cases}$.

We say that a point x of X is *ranked* if $\mathrm{CB}(x) = \max\{\beta \mid x \in \Gamma^\beta(X)\}$ is an ordinal.

b) Show that there must be some ordinal α_0 with $\Gamma^{\alpha_0}(X) = \Gamma^{\alpha_0+1}(X)$. Deduce that $\Gamma^{\alpha_0}(X) = \Gamma^\gamma(X)$ for all $\gamma \geq \alpha_0$.

c) Suppose that $\Gamma^\alpha(X) \neq \emptyset = \Gamma^{\alpha+1}(X)$. Show that $\Gamma^\alpha(X)$ is finite.

d) Show that if every point of X is ranked, then the smallest α with $\Gamma^\alpha(X) = \Gamma^{\alpha+1}(X)$ is either 0 or a successor ordinal.

   Hint: Compactness.

e) Let now $\mathcal{M}$ be a a structure with $(x \doteq x)$ minimal in $\mathcal{M}$ and consider the space of 1-types $X = \mathrm{S}_1(A)$ over a subset $A \subset M$. Compute $\Gamma^\alpha(X)$ for every ordinal α.

f) Is every type in $\mathrm{S}_1(A)$ ranked?

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.