Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Übungen: Dr. Francesco Gallinaro — Sommersemester 2023

Modelltheorie II
Homework Sheet 11
Deadline: 17.07.2023, 14 Uhr
Last Homework Sheet

Unless explicitly mentioned, we work inside a sufficiently large saturated model $\mathbb{U}$ of an ω-stable complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

Exercise 1 (10 Points)**.**

a) Let p and q be stationary types over a subset of parameters C of $\mathbb{U}$. A pair (a, b) with a realizing p and b realizing q is an *independent pair* if $a \mathop{\smile\!\!\!\!\!\mid}_C b$. Show that any two independent pairs have the same type over C. In particular, the type of an independent pair is unique (and we denote it by $p \otimes q$).

b) Given now a subset $C \subset D \subset \mathbb{U}$, consider p', resp. q', the unique non-forking extension of p, resp. q, to D. Show that $p' \otimes q'$ is the unique non-forking extension of $p \otimes q$ to D. In particular, the type $p \otimes q$ is stationary.

Consider now two definable groups G and H without parameters. An *independent pair* for $G \times H$ over the subset A of parameters is a pair (g, h) in $G \times H$ realizing $p \otimes q$, where p is the generic type of G^0 over A and q is the generic type of H^0 over A.

c) Given an independent pair (g, h) of $G \times H$, show using (a) that $G^0 \times H^0 \subset \mathrm{Stab}(g, h)$.

d) Deduce that the independent pair (g, h) over A is a generic element of $G \times H$ over A.

e) Conclude that $G \times H$ is connected if both G and H are.

Exercise 2 (10 Points)**.** Consider a definable group G without parameters. A non-empty subset X of G (possibly non-definable) is *indecomposable* if, whenever $H \leq G$ is a definable subgroup (with parameters) such that X intersects non-trivially two different cosets of H, then X intersects non-trivially infinitely many cosets of H.

a) Determine explicitly all indecomposable subsets of the multiplicative group $\mathbb{C}^*$ (in the strongly minimal theory ACF_0). Deduce from this description that a non-empty subset of an indecomposable subset need not be indecomposable.

b) Show that a definable subgroup of G is indecomposable if and only if it is connected.

A non-empty subset X *admits an irredundant indecomposable decomposition* if $X = X_1 \cup \ldots \cup X_n$, where each $X_i \neq \emptyset$ is indecomposable and no subunion $\bigcup_{j \in J} X_{i_j}$, with $|J| \geq 2$, is indecomposable.

c) Show that an irredundant indecomposable decomposition $X = X_1 \cup \ldots \cup X_n$ is a disjoint union (so $X_i \cap X_j = \emptyset$ for all $i \neq j$).

   Hint: If Y and Z are both indecomposable with $Y \cap Z \neq \emptyset$, what do we conclude about $Y \cup Z$?

d) Show that any two irredundant indecomposable decompositions of X are equal up to permutation, that is, if $X = X_1 \cup \ldots \cup X_n$ and $X = Y_1 \cup \cdots \cup Y_m$ are irredundant indecomposable decompositions, then $n = m$ and for each i we have $X_i = Y_{\tau(i)}$ for some τ in S_n.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im Fach 3.07 im Keller des mathematischen Instituts.