Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Übungen: Dr. Francesco Gallinaro — Sommersemester 2023

Modelltheorie II

Homework Sheet 10

Deadline: 10.07.2023, 14 Uhr

Unless explicitly mentioned, we consider a definable group G without parameters inside a sufficiently large saturated model $\mathbb{U}$ of an ω-stable complete first-order theory T with infinite models in a fixed language $\mathcal{L}$.

Exercise 1 (6 Points)**.**

a) Let A be a subset of parameters of $\mathbb{U}$. Show that for every A-definable subset X of G, there exists a finite subset F of G such that $G = F \cdot X = \bigcup_{g\in F} g \cdot X$ or $G = F \cdot (G \setminus X)$.

b) Consider now the collection of formulae

$$\Sigma(x) = \{x \in G\} \cup \{\neg\varphi[x, a] \mid \text{ for no finite } F \text{ in } G \text{ we have that } G = F \cdot \varphi[\mathbb{U}, a]\}_{a\in A}.$$

Show that Σ is finitely consistent. How many completions does Σ have?

c) Consider now the additive group of (a sufficiently saturated extension of) the field of real numbers and set $A = \mathbb{Q}$. Does the conclusion of (a) hold for this structure?

Exercise 2 (10 Points)**.** Let $\mathcal{M}$ be an elementary substructure of $\mathbb{U}$ and choose some type p in the type space $S_G(M)$.

a) Assume that there is an M-definable subgroup H of G and some element m in M such that p contains the formula "$x \in H \cdot m$". Show that $\mathrm{Stab}(p) \leq H$.

b) Assume now that $\mathrm{RM}(p) = \mathrm{RM}(\mathrm{Stab}(p))$. Deduce from the above that the type p contains the formula "$x \in \mathrm{Stab}(p)^0 \cdot m$" for some m in M.

   Hint: Redo the proof of $\mathrm{RM}(\mathrm{Stab}(p)) \leq \mathrm{RM}(p)$ and use Exercise 1 of the Homework Sheet 7.

c) Deduce from the above that $\mathrm{Stab}(p)$ is connected and p is the unique generic type of some M-definable coset of its stabilizer, whenever $\mathrm{RM}(p) = \mathrm{RM}(\mathrm{Stab}(p))$.

d) Suppose now that p_0 is a type over some subset $A = \mathrm{acl}^{eq}(A) \subset M$ such that $\mathrm{RM}(p_0) = \mathrm{RM}(\mathrm{Stab}(p_0))$. Conclude that $\mathrm{Stab}(p_0)$ is connected and p_0 is the unique generic type of some A-definable coset of its stabilizer,

   Hint: Exercise 2 (a) of the Homework Sheet 7.

Exercise 3 (4 Points)**.** A structure $\mathcal{N}$ in some first-order countable language $\mathcal{L}_0$ is *definable* in $\mathbb{U}$ with parameters in A if there exists some $\mathcal{L}$-definable subset X of some cartesian product of $\mathbb{U}$ with parameters in A such that $N = X(\mathbb{U})$. Furthermore, we require that for every function (resp. relation) symbol f (resp. R) in $\mathcal{L}_0$ there is a subset $\mathrm{Def}(f)$ (resp. $\mathrm{Def}(R)$) which is $\mathcal{L}$-definable over A such that for every $a_1, \ldots, a_n$ and b from N, we have

$$\mathcal{N} \models f(a_1, \ldots, a_n) = b \iff \mathbb{U} \models (a_1, \ldots, a_n, b) \in \mathrm{Def}(f),$$

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resp.

$$\mathcal{N} \models R(a_1, \ldots, a_n) \iff \mathbb{U} \models (a_1, \ldots, a_n) \in \mathrm{Def}(R).$$

a) Show that there exists a countable subset of parameters C of $\mathbb{U}$ such that for every $\mathcal{L}_0$-formula $\varphi[x_1, \ldots, x_n]$ there is an instance $\psi_\varphi[x_1, \ldots, x_n, d]$ of an $\mathcal{L}$-formula, with d a tuple from $C \cup A$, such that for all $a_1, \ldots, a_n$ from N we have

$$\mathcal{N} \models \varphi[a_1, \ldots, a_n] \iff \mathbb{U} \models \psi_\varphi[a_1, \ldots, a_n, d].$$

What does C consist of?

Hint: Induction on the complexity of φ.

b) Deduce that the $\mathcal{L}_0$-theory $\mathrm{Th}(\mathcal{N})$ is ω-stable.